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A strain-difference-based nonlocal elasticity model

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Abstract

A two-component local/nonlocal constitutive model for (macroscopically) inhomogeneous linear elastic materials (but constant internal length) is proposed, in which the stress is the sum of the local stress and a nonlocal-type stress expressed in terms of the strain difference field, hence identically vanishing in the case of uniform strain. Attention is focused upon the particular case of piecewise homogeneous material. The proposed model is thermodynamically consistent with a suitable free energy potential. It constitutes an improved form of the Vermeer and Brinkgreve [A new effective nonlocal strain measure for softening plasticity. In: Chambon, R., Desrues, J., Vardulakis, I. (Eds.), Localization and Bifurcation theory for Soils and Rocks. Balkema, Rotterdam, 1994, pp. 89–100] model, and can also be considered derivable from the Eringen nonlocal elasticity model through a suitable enhancement technique based on the concept of redistribution of the local stress. The concept of *equivalent distance* is introduced to macroscopically account for the further attenuation effects produced by the inhomogeneity upon the long distance interaction forces. With the aid of a piecewise homogeneous bar in tension, a portion of which degrades progressively till failure, it is shown that—under a suitable choice of a material constant—the solution procedure exhibits no pathological features (numerical instability, mesh sensitivity) in every degraded bar condition, including the limit idealized stress-free condition of the failed bar.

Keywords: Nonlocal elasticity; Inhomogeneous materials; Boundary surface effects

1. Introduction

In the framework of nonlocal (integral type) elasticity (Kröner, 1967; Edelen and Laws, 1971; Eringen and Edelen, 1972; Rogula, 1982), Eringen and co-workers (Eringen and Kim, 1974; Eringen et al., 1977; Eringen, 1978; Eringen, 1979; Eringen, 1987) proposed a simplified theory for linear (macroscopically) homogeneous isotropic elasticity, which differs from the classical one in the stress–strain constitutive relation only, whereas the equilibrium and compatibility equations remain unaltered. In this way, the

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mentioned authors were able to show that the crack-tip stress singularity, predicted by classical elasticity, does not arise making use of the nonlocal theory (see also Artan and Yelkenci, 1996; Zhou et al., 1999).

Numerous subsequent studies were devoted to the Eringen nonlocal elasticity model. Altan (1989a) and Altan (1989b) addressed questions as the existence and uniqueness of the elasticity boundary-value problem solution; Wang and Dhaliwal (1993a) formulated a reciprocity relation for nonlocal polar continua. Similar questions were also addressed by Altan (1991) and Wang and Dhaliwal (1993b) within nonlocal thermo-elasticity. Polizzotto (2001) extended the classical variational principles to nonlocal elasticity. Bažant and Jirásek (2002) reviewed nonlocal elasticity and in particular the applications of the Eringen model and of its enhanced versions to plasticity and damage mechanics. Pisano and Fuschi (2003) provided a closed form solution for a nonlocal elastic bar in tension. Jirásek and Rolshoven (2003) made a critical comparison of the existing integral models in the domain of plasticity and softening materials.

A phenomenological nonlocal-type elasticity stress–strain law is as a rule required to satisfy the basic requisite to provide a uniform stress field whenever the strain field is uniform, just like in the case of local-type (homogeneous) elastic material. Such a material model proves to be unable to capture surface physics phenomena, that is, the complex modifications that the microstructure undergoes in the vicinity of the boundary surface (boundary surface microscale effects), which however are usually considered (and are considered here) secondary for a phenomenological description of the constitutive law.

The previously mentioned Eringen model satisfies the above requisite in the case of infinite domain, but not in the case of finite domain due to the boundary surface *macroscopic* effects (the finite support of the influence function exceeds the integration domain). The latter effects are in general accompanied by numerical instability and mesh sensitivity in the solution of boundary-value problems.

Various enhancement rules have been adopted in the literature to render Eringen model capable to preserve corresponding uniform stress/strain fields (Bažant and Jirásek, 2002; Jirásek and Rolshoven, 2003). The most popular is that suggested by Pijaudier-Cabot and Bažant (1987) in the context of damage mechanics, consisting in rescaling the influence function, which then loses symmetry.

The symmetry of the influence function constitutes an important pre-requisite for computational, as well as for theoretical reasons (beneficial influences on material stability, existence of extremum principles), (Bažant and Jirásek, 2002; Polizzotto, 2001). A symmetry-preserving enhancement rule, proposed by Polizzotto (2002) and by Borino et al. (2002) and Borino et al. (2003), consists in the addition of a suitable inhomogeneous local constitutive component to the nonlocal one in the vicinity of the boundary surface. This two-component local/nonlocal constitutive model differs from analogous models of the literature, which are characterized by the homogeneity of the added local constitutive component (Bažant and Chang, 1984; Altan, 1989a; Vermeer and Brinkgreve, 1994; Strömberg and Ristinmaa, 1996) (see also Bažant and Planas, 1998; Jirásek and Rolshoven, 2003). The stabilizing effects induced by the addition of this local component were studied by Bažant and Chang (1984) through the positivity of the Fourier transform of the modified kernel.

The purpose of the present paper is to propose a nonlocal elasticity model for (macroscopically) inhomogeneous materials (but constant internal length), in which the stress is the superposition of two contributions, one coinciding with the local stress, the other is of nonlocal nature, given by a weighing formula operating on the strain difference. The model exhibits the natural capacity to predict, for a uniform strain field, the related local stress field. In addition to the elastic moduli and the internal length parameter, ℓ, the model contains a material constant that controls the volumetric proportions of the local and nonlocal constitutive components, and that, in numerical simulations, can be chosen such as to guarantee stability of the computational procedure. The proposed model constitutes an improvement of the Vermeer and Brinkgreve (1994) model, with which it coincides within the core domain, that is at points where the boundary surface influence is vanishing.

For greater completeness, the Eringen nonlocal elasticity model is revised in Section 2 together with some enhanced forms of it. Then, a simpler form of the proposed model is presented in Section 3, suitable to the case of (macroscopically) homogeneous material. In this version, the proposed model includes as a particular case the previously mentioned two-component local/nonlocal constitutive model (Polizzotto, 2002; Borino et al., 2002; Borino et al., 2003); its similarities with the Vermeer–Brinkgreve model are also recognized. The related total strain energy is computed and the relevant expression of the Helmholtz free energy is utilized in order to verify the thermodynamic consistency of the constitutive equation. In Section 4, a suitable form of the free energy potential is used as a starting point for extending the proposed model to the case of a (macroscopically) inhomogeneous material. In particular, the case of piecewise homogeneous material is addressed with two or more subdomains. In Section 5, it is argued that inhomogeneity is the cause of an additional attenuation of the long distance interaction forces between particles located on the opposite sides with respect to an internal boundary. The concept of equivalent distance is introduced as a means to account for the augmented attenuation effects. In Section 6 a numerical application to a bar structure is reported. The conclusions are drawn in Section 7.

1.1. Notation

A compact notation is used, with bold-face letters for vectors and tensors. The 'dot' and 'colon' products between vectors and tensors denote the simple and double index contraction operations, respectively, for instance: $\mathbf{u}\cdot\mathbf{v} = u_i v_i$, $\boldsymbol{\sigma}:\boldsymbol{\varepsilon} = \sigma_{ij}, \varepsilon_{ij}$, $\boldsymbol{\sigma}\cdot\mathbf{n} = \{\sigma_{ji}, n_j\}$, $\mathbf{D}:\boldsymbol{\varepsilon} = \{D_{khij}\varepsilon_{kh}\}$. The subscripts denote Cartesian components and the repeated index summation rule is to be applied. Cartesian orthogonal co-ordinates $\mathbf{x} = (x_1, x_2, x_3)$ are employed. The symbol $:=$ means equality by definition. Other symbols will be defined in the text where they appear for the first time.

2. The Eringen nonlocal elasticity model

For completeness sake, the Eringen model is briefly reviewed in this section, and a few enhanced forms of it are reminded. In the Eringen model (Eringen and Kim, 1974; Eringen et al., 1977; Eringen, 1987), the stress–strain relation has the form

$$\boldsymbol{\sigma}(\mathbf{x}) = \mathbf{D}_0 : \int_V g(\mathbf{x}, \mathbf{x}'), \boldsymbol{\varepsilon}(\mathbf{x}')\,\mathrm{d}V', \tag{1}$$

where V is the domain occupied by the material, $\mathrm{d}V' := \mathrm{d}V(\mathbf{x}')$, $\mathbf{D}_0$ is the usual moduli fourth-order tensor of local (homogeneous) elasticity and $g(\mathbf{x}, \mathbf{x}') = g(\mathbf{x}', \mathbf{x})$ is the influence, or attenuation, function. g is a (continuous) function of the Euclidean distance, $r = \|\, \mathbf{x}' - \mathbf{x} \,\|$, that is $g = \bar{g}(r/\ell)$, where ℓ is the internal length parameter; it is a positive more or less rapidly decreasing function, such that $g \simeq 0$ for $r \geqslant R$, where R is the influence distance, radius of the influence sphere $\Sigma(\mathbf{x})$ centered at $\mathbf{x}$, in general much smaller than the smallest linear dimension of V. As a rule, g will be taken vanishing for $r > R$ in the following.

g satisfies the normalization condition (obviously independent of $\mathbf{x}$):

$$\int_{V_\infty} g(\mathbf{x}, \mathbf{x}')\,\mathrm{d}V' = \int_{\Sigma(\mathrm{x})} g(\mathbf{x}, \mathbf{x}')\,\mathrm{d}V' = 1, \tag{2}$$

where V_∞ is the (convex) infinite domain in which V is embedded. Eq. (2) guarantees that $g(\mathbf{x}, \mathbf{x}') \to \delta(\mathbf{x}' - \mathbf{x})$ (Dirac delta) for $\ell \to 0$, that is, the Eringen model recovers local elasticity for $\ell \to 0$.

Several mathematical forms have been suggested for g, like the error function, the bell-shaped function and the bi-exponential function, (see Eringen, 1987; Bažant and Chang, 1984; Polizzotto, 2001; Bažant and Jirásek, 2002). The bi-exponential influence function, having the form $g = (1/2\ell)\exp(-r/\ell)$ for $r < R$, but $g = 0$ for $r > R = 6\ell$, in a one-dimensional setting, will be used in all applications to follow.

The concept of *geodetical distance*, $r(\mathbf{x}, \mathbf{x}')$, has been suggested by Polizzotto (2001) as the length of the shortest path joining $\mathbf{x}$ with $\mathbf{x}'$ without intersecting the boundary surface, to be used in place of the Euclidean distance when an obstacle as hole, or crack, is located between $\mathbf{x}$ and $\mathbf{x}'$, or more in general the domain V is not convex (see also Peerlings et al., 2001).

The (adimensional) function

$$\gamma(\mathbf{x}) := \int_V g(\mathbf{x}, \mathbf{x}')\,\mathrm{d}V', \tag{3}$$

which represents the nonlocal counterpart of a unit uniform field in V, is continuous and bounded, $0 < \gamma(\mathbf{x}) \leqslant 1$. For the influence functions of common use, $\gamma(\mathbf{x})$ admits a plot like that of Fig. 1(b), which refers to the bi-exponential influence function in a one-dimensional domain, Fig. 1(a), for which $0.5 \leqslant \gamma(x) \leqslant 1$.

On taking $\boldsymbol{\varepsilon}(\mathbf{x})$ uniform in V, say $\boldsymbol{\varepsilon}(\mathbf{x}) = \bar{\boldsymbol{\varepsilon}}$, Eq. (1) gives

$$\boldsymbol{\sigma}(\mathbf{x}) = \gamma(\mathbf{x})\mathbf{D}_0 : \bar{\boldsymbol{\varepsilon}}, \tag{4}$$

where the function $\gamma(\mathbf{x})$ interprets the amount and distribution of the mentioned boundary surface macroscopic effects that make $\boldsymbol{\sigma}(\mathbf{x})$ nonuniform in (4). These effects turn out to be vanishing at all points $\mathbf{x} \in V_c \subset V$, ($V_c$ = core domain), where $\Sigma(\mathbf{x})$ does not exceed ∂V, hence by (2) $\gamma(\mathbf{x}) = 1$; but they are nonvanishing at points $\mathbf{x}$ closer to ∂V, where $\Sigma(\mathbf{x})$ exceeds the boundary surface, hence by (2) $\gamma(\mathbf{x}) < 1$.

From (4) follows that the Eringen model (1) does not comply with the basic requisite usually required for a phenomenological nonlocal model, that is, to provide uniform stress in the presence of uniform strain. As a remedy to this drawback, Pijaudier-Cabot and Bažant (1987) suggested to replace $g(\mathbf{x}, \mathbf{x}')$ of (1) with the rescaled function

$$g^*(\mathbf{x}, \mathbf{x}') = g(\mathbf{x}, \mathbf{x}')/\gamma(\mathbf{x}), \tag{5}$$

which however is not symmetric.

An alternative remedy saving the symmetry was suggested by Polizzotto (2002) on the basis of long distance energy redistribution considerations. More simply, one can observe that the integrand of (1), rewritten in the equivalent form

$$\mathbf{s}(\mathbf{x}', \mathbf{x}) := g(\mathbf{x}', \mathbf{x})\mathbf{D}_0 : \boldsymbol{\varepsilon}(\mathbf{x}), \tag{6}$$

represents the specific nonlocal stress induced in $\mathbf{x}'$ by the strain $\boldsymbol{\varepsilon}(\mathbf{x})$ (the adjective “specific” is used to signify that $\mathbf{s}(\mathbf{x}', \mathbf{x})$ has the dimension of a stress divided by a volume); also, by (3), the integral of (6) with respect to $\mathbf{x}' \in V$ is

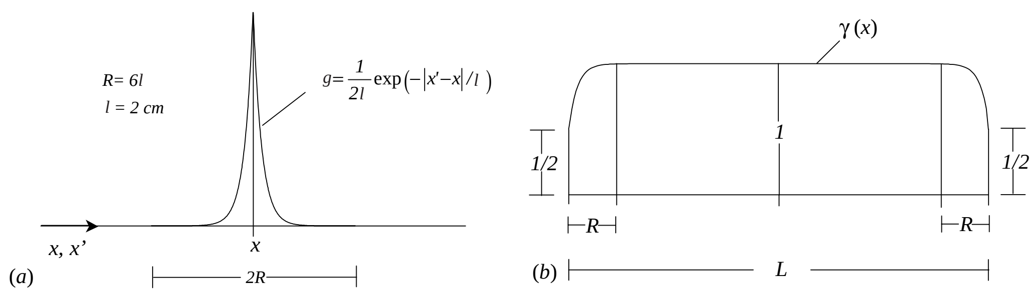

Fig. 1. Plots representing: (a) the bi-exponential influence function in one-dimensional (infinite) domain, $g = (1/2\ell)\exp(-\mid x' - x \mid/\ell)$, $\ell = 2$ cm, $R = 12$ cm; (b) the related function $\gamma(x)$, Eq. (3), for $0 \leqslant x \leqslant L$.

$$\int_V \mathbf{s}(\mathbf{x}', \mathbf{x})\, \mathrm{d}V' = \gamma(\mathbf{x})\mathbf{D}_0 : \boldsymbol{\varepsilon}(\mathbf{x}), \tag{7}$$

which coincides with the local stress $\mathbf{D}_0 : \boldsymbol{\varepsilon}(\mathbf{x})$ in the core domain V_c, where $\gamma(\mathbf{x}) = 1$. This result can be macroscopically interpreted stating the following: in an idealized process leading from the local to the nonlocal constitutive model, the local stress $\mathbf{D}_0 : \boldsymbol{\varepsilon}(\mathbf{x})$ redistributes completely within the influence sphere at points $\mathbf{x} \in V_c$, but only in part at points $\mathbf{x}$ closer to the boundary ∂V where $\gamma(\mathbf{x}) < 1$ and where thus the nonredistributed local stress, $[1 - \gamma(\mathbf{x})]\mathbf{D}_0 : \boldsymbol{\varepsilon}(\mathbf{x})$, there remains attached to $\mathbf{x}$. In other words, Eq. (1) is to be considered the right phenomenological stress–strain law in the case of infinite domain, but in the case of finite domain is to be replaced by the *enhanced* law:

$$\boldsymbol{\sigma}(\mathbf{x}) = [1 - \gamma(\mathbf{x})]\mathbf{D}_0 : \boldsymbol{\varepsilon}(\mathbf{x}) + \mathbf{D}_0 : \int_V g(\mathbf{x}, \mathbf{x}')\boldsymbol{\varepsilon}(\mathbf{x}')\mathrm{d}V', \tag{8}$$

which accounts for the mentioned boundary surface macroscopic effects. The enhanced law (8) distinguishes from (1) for the added inhomogeneous local part, characterized by the moduli tensor $\mathbf{D}(\mathbf{x}) = [1 - \gamma(\mathbf{x})]\mathbf{D}_0$, which is nonvanishing in the boundary layer, $V \setminus V_c$, but vanishing in the core domain V_c.

Independently, Borino et al. (2002) and Borino et al. (2003) adopted an integral formula similar to (8) within the context of damage mechanics.

3. Strain-difference-based nonlocal model: homogeneous material

Continuing to have in mind a (macroscopically) homogeneous elastic material, an alternative to the Eringen model (1) is here proposed in the form

$$\boldsymbol{\sigma}(\mathbf{x}) = \mathbf{D}_0 : \boldsymbol{\varepsilon}(\mathbf{x}) + \alpha\mathbf{D}_0 : \int_V G(\mathbf{x}, \mathbf{x}')[\boldsymbol{\varepsilon}(\mathbf{x}') - \boldsymbol{\varepsilon}(\mathbf{x})]\, \mathrm{d}V', \tag{9}$$

where α is an adimensional scalar constant and $G(\mathbf{x}, \mathbf{x}')$ is an influence function satisfying (2). This model belongs to the class of two-component local/nonlocal models; it grounds on the basic idea by which the nonlocal stress is expressed by the addition to the local stress of a nonlocal stress depending on the strain difference field, such that (9) naturally gives uniform stress for uniform strain for whatever domain V.

With the position:

$$\Gamma(\mathbf{x}) := \int_V G(\mathbf{x}, \mathbf{x}')\, \mathrm{d}V', \tag{10}$$

Eq. (9) can also be rewritten as

$$\boldsymbol{\sigma}(\mathbf{x}) = [1 - \alpha\Gamma(\mathbf{x})]\mathbf{D}_0 : \boldsymbol{\varepsilon}(\mathbf{x}) + \alpha\mathbf{D}_0 : \int_V G(\mathbf{x}, \mathbf{x}')\boldsymbol{\varepsilon}(\mathbf{x}')\, \mathrm{d}V', \tag{11}$$

which coincides with (8) if $G(\mathbf{x}, \mathbf{x}') \equiv g(\mathbf{x}, \mathbf{x}')$ and $\alpha = 1$. This implies that the proposed strain-difference-based model can also be considered derivable from the Eringen model (1) through the previously introduced symmetry-saving enhancement procedure (by which (8) has been derived from (9)).

Since $\Gamma(\mathbf{x}) = 1$ in the core domain V_c related to G, follows that, in this V_c, Eq. (11) simplifies as

$$\boldsymbol{\sigma}(\mathbf{x}) = \mathbf{D}_0 : \left[(1 - \alpha)\boldsymbol{\varepsilon}(\mathbf{x}) + \alpha \int_V G(\mathbf{x}, \mathbf{x}')\boldsymbol{\varepsilon}(\mathbf{x}')\, \mathrm{d}V'\right], \tag{12}$$

which is formally as the Vermeer and Brinkgreve (1994) model of common use in plasticity as an effective strain localization limiter (see e.g. Jirásek and Rolshoven, 2003). However, the latter model does not satisfy the requisite to provide uniform stress under uniform strain, (unless the rescaled influence function (5) is employed, but then symmetry is lost), hence (11) can be viewed as an improved version of the mentioned Vermeer–Brinkgreve model.

It is worth noting that the total strain energy associated to (11), or (9), can be written in the form:

$$W = \frac{1}{2}\int_V \boldsymbol{\varepsilon}(\mathbf{x}) : [1 - \alpha\Gamma(\mathbf{x})]\mathbf{D}_0 : \boldsymbol{\varepsilon}(\mathbf{x})\,\mathrm{d}V + \frac{\alpha}{2}\int_V\int_V \boldsymbol{\varepsilon}(\mathbf{x}) : G(\mathbf{x},\mathbf{x}')\mathbf{D}_0 : \boldsymbol{\varepsilon}(\mathbf{x}')\,\mathrm{d}V'\,\mathrm{d}V, \tag{13}$$

which is assumed positive definite (Polizzotto, 2001; Bažant and Jirásek, 2002). Another form for (13), useful for the developments to follow, can be derived by introducing a (symmetric) *primitive influence function*, say $g(\mathbf{x},\mathbf{x}')$, such that

$$G(\mathbf{x},\mathbf{x}') = \int_V g(\mathbf{x},\mathbf{z})g(\mathbf{z},\mathbf{x}')\,\mathrm{d}V(\mathbf{z}) \quad \forall \mathbf{x},\mathbf{x}' \in V; \tag{14}$$

that is, $G(\mathbf{x},\mathbf{x}')$ represents the (symmetric) *weighed influence function* associated to $g(\mathbf{x},\mathbf{x}')$. It can be verified that, if $g(\mathbf{x},\mathbf{x}')$ is an influence function satisfying (2), then $G(\mathbf{x},\mathbf{x}')$ given by (14) also satisfies (2). At points $\mathbf{x} \in V$ sufficiently far from the boundary ∂V, $G(\mathbf{x},\mathbf{x}')$ as function of $\mathbf{x}'$ exhibits a fixed shape like $g(\mathbf{x},\mathbf{x}')$ but possesses a greater influence distance, as shown in Fig. 2(a), whereas is affected by the boundary surface ∂V at points $\mathbf{x}$ closer to the latter, Fig. 2(b).

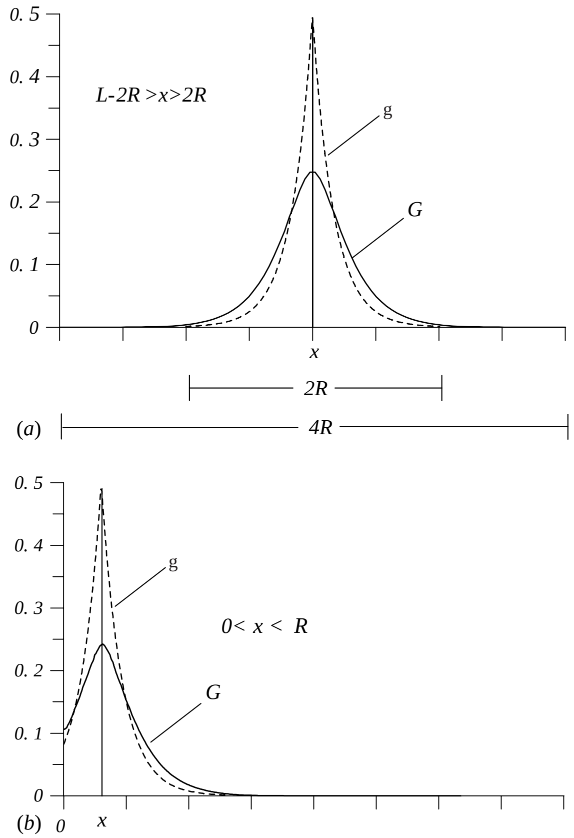

Fig. 2. Weighed influence function $G(r/\ell)$ (solid line) compared with the bi-exponential influence function, $g(r/\ell) = (1/2\ell)\exp(-r/\ell)$ (solid line), with $\ell = 1$ cm, $R = 6$ cm, in a one-dimensional domain $0 \leqslant x \leqslant L$, $(L \gg R)$: (a) at a point x far from the ends; (b) at a point x close to one of the ends.

Let the *nonlocal strain*, $\widehat{\boldsymbol{\varepsilon}}(\mathbf{x})$, be defined by using the primitive influence function $g(\mathbf{x},\mathbf{x}')$, that is,

$$\widehat{\boldsymbol{\varepsilon}}(\mathbf{x}) = \mathscr{R}(\boldsymbol{\varepsilon})(\mathbf{x}) := \int_V g(\mathbf{x},\mathbf{x}')\boldsymbol{\varepsilon}(\mathbf{x}')\,\mathrm{d}V'. \tag{15}$$

Then, substituting (14) in (13), one easily obtains

$$W = \frac{1}{2}\int_V \{\boldsymbol{\varepsilon}(\mathbf{x}) : [1-\alpha\Gamma(\mathbf{x})]\mathbf{D}_0 : \boldsymbol{\varepsilon}(\mathbf{x}) + \widehat{\boldsymbol{\varepsilon}}(\mathbf{x}) : \alpha\mathbf{D}_0 : \widehat{\boldsymbol{\varepsilon}}(\mathbf{x})\}\,\mathrm{d}V \tag{16}$$

the integrand of which can be identified with the functional

$$\psi(\boldsymbol{\varepsilon},\mathscr{R}(\boldsymbol{\varepsilon}),\mathbf{x}) := \frac{1}{2}\boldsymbol{\varepsilon} : [1-\alpha\Gamma(\mathbf{x})]\mathbf{D}_0 : \boldsymbol{\varepsilon} + \frac{1}{2}\mathscr{R}(\boldsymbol{\varepsilon}) : \alpha\mathbf{D}_0 : \mathscr{R}(\boldsymbol{\varepsilon}) \tag{17}$$

representing the relevant Helmholtz free energy density at the generic point $\mathbf{x}\in V$. Following Polizzotto (2001) and observing that the integral operator $\mathscr{R}$ is self-adjoint because of the symmetry of the kernel $g(\mathbf{x},\mathbf{x}')$, the thermodynamically consistent stress corresponding to (16) can be found to be (*state equation*):

$$\boldsymbol{\sigma} = \frac{\partial\psi}{\partial\boldsymbol{\varepsilon}} + \mathscr{R}\left(\frac{\partial\psi}{\partial\mathscr{R}(\boldsymbol{\varepsilon})}\right) \quad \text{in } V, \tag{18}$$

which can be shown to coincide with (11), or (9). This means that the stress (9), or (11), is thermodynamically consistent with the free energy potential (17), provided $G(\mathbf{x},\mathbf{x}')$ complies with (14).

Note that the knowledge of the primitive influence function, $g(\mathbf{x},\mathbf{x}')$, is not required by the stress–strain law (9), or (11), hence the weighed influence function, $G(\mathbf{x},\mathbf{x}') = \mathscr{R}(g)(\mathbf{x},\mathbf{x}')$, Eq. (14), can be there specified independently, but then the thermodynamic consistency with (17) is generally lost.

4. Strain-difference-based nonlocal model: inhomogeneous material

In this section, an inhomogeneous nonlocal elastic material (but with constant internal length) occupying the domain V is considered, which is characterized by a local moduli tensor, $\mathbf{D}(\mathbf{x})$, symmetric and positive definite everywhere in V. The material inhomogeneity is conjectured to constitute the source of further attenuation effects on the long distance particles interaction forces. Since no experimental indication seems to be available, the hypothesis is here introduced that the increased attenuation effects may still be accounted for by means of a symmetric attenuation function $g(\mathbf{x},\mathbf{x}') = \bar{g}(r/\ell)$, but using, in place of r, a suitably incremented *equivalent distance*, $r_{\mathrm{eq}} = r_{\mathrm{eq}}(\mathbf{x},\mathbf{x}') > r(\mathbf{x},\mathbf{x}')$, (see Section 5).

By extrapolation of the homogeneous material case, let the free energy ψ be taken in the following form:

$$\psi(\boldsymbol{\varepsilon},\mathscr{R}(\boldsymbol{\varepsilon}),\mathbf{x}) := \frac{1}{2}\boldsymbol{\varepsilon} : \mathbf{D}^{(0)}(\mathbf{x}) : \boldsymbol{\varepsilon} + \frac{\alpha}{2}\mathscr{R}(\boldsymbol{\varepsilon}) : \mathbf{D}(\mathbf{x}) : \mathscr{R}(\boldsymbol{\varepsilon}), \tag{19}$$

where $\mathbf{D}^{(0)}(\mathbf{x})$ is an unknown local-type moduli tensor and α is some scalar constant. By (18) and using the notation $\widehat{\boldsymbol{\varepsilon}} = \mathscr{R}(\boldsymbol{\varepsilon})$ of (15), one has

$$\boldsymbol{\sigma}(\mathbf{x}) = \mathbf{D}^{(0)}(\mathbf{x}) : \boldsymbol{\varepsilon}(\mathbf{x}) + \alpha\int_V g(\mathbf{x},\mathbf{z})\mathbf{D}(\mathbf{z}) : \widehat{\boldsymbol{\varepsilon}}(\mathbf{z})\,\mathrm{d}V(\mathbf{z}), \tag{20}$$

which, remembering the integral expression of $\widehat{\boldsymbol{\varepsilon}}$ in (15), can also be written

$$\boldsymbol{\sigma}(\mathbf{x}) = \mathbf{D}^{(0)}(\mathbf{x}) : \boldsymbol{\varepsilon}(\mathbf{x}) + \alpha\int_V \mathbf{k}(\mathbf{x},\mathbf{x}') : \boldsymbol{\varepsilon}(\mathbf{x}')\,\mathrm{d}V', \tag{21}$$

where $\mathbf{k}(\mathbf{x},\mathbf{x}')$ denotes the material *nonlocal moduli tensor*, defined as

$$\mathbf{k}(\mathbf{x},\mathbf{x}') := \int_V \mathbf{D}(\mathbf{z}) g(\mathbf{z},\mathbf{x}) g(\mathbf{z},\mathbf{x}')\,\mathrm{d}V(\mathbf{z}), \tag{22}$$

where $g(\mathbf{z},\mathbf{x}) = \bar{g}(r_{\mathrm{eq}}(\mathbf{z},\mathbf{x})/\ell)$ with $r_{\mathrm{eq}}(\mathbf{z},\mathbf{x})$ to be specified later on (Section 5).

On the other hand, since (21) must give $\boldsymbol{\sigma}(\mathbf{x}) = \mathbf{D}(\mathbf{x}) : \bar{\boldsymbol{\varepsilon}}$ for any uniform strain field $\bar{\boldsymbol{\varepsilon}}$, one easily obtains from (21):

$$\mathbf{D}^{(0)}(\mathbf{x}) = \mathbf{D}(\mathbf{x}) - \alpha\mathbf{K}(\mathbf{x}) \quad \forall \mathbf{x} \in V, \tag{23}$$

where

$$\mathbf{K}(\mathbf{x}) := \int_V \mathbf{k}(\mathbf{x},\mathbf{x}')\,\mathrm{d}V'. \tag{24}$$

Therefore, the stress–strain relation (21) takes on the form:

$$\boldsymbol{\sigma}(\mathbf{x}) = [\mathbf{D}(\mathbf{x}) - \alpha\mathbf{K}(\mathbf{x})] : \boldsymbol{\varepsilon}(\mathbf{x}) + \alpha \int_V \mathbf{k}(\mathbf{x},\mathbf{x}') : \boldsymbol{\varepsilon}(\mathbf{x}')\,\mathrm{d}V' \tag{25}$$

or, equivalently,

$$\boldsymbol{\sigma}(\mathbf{x}) = \mathbf{D}(\mathbf{x}) : \boldsymbol{\varepsilon}(\mathbf{x}) + \alpha \int_V \mathbf{k}(\mathbf{x},\mathbf{x}') : [\boldsymbol{\varepsilon}(\mathbf{x}') - \boldsymbol{\varepsilon}(\mathbf{x})]\mathrm{d}V'. \tag{26}$$

This nonlocal stress–strain relation is the extension of (9) to inhomogeneous material.

For practical reasons, it is of interest to consider a piecewise homogeneous material domain, and in particular the case $V = V_A \cup V_B$, with V_A and V_B each homogeneous, such that

$$\mathbf{D}(\mathbf{x}) = \mathbf{D}_A \text{ in } V_A, \quad \mathbf{D}(\mathbf{x}) = \mathbf{D}_B \text{ in } V_B, \tag{27}$$

hence the internal boundary S_{AB} that separates V_A and V_B from each other constitutes a discontinuity surface for $\mathbf{D}(\mathbf{x})$. Therefore, noting that $\mathbf{k}(\mathbf{x},\mathbf{x}')$ correspondingly takes on the form

$$\mathbf{k}(\mathbf{x},\mathbf{x}') = \mathbf{D}_A G_A(\mathbf{x},\mathbf{x}') + \mathbf{D}_B G_B(\mathbf{x},\mathbf{x}'), \tag{28}$$

where

$$G_I(\mathbf{x},\mathbf{x}') := \int_{V_I} g(\mathbf{x},\mathbf{z}) g(\mathbf{z},\mathbf{x}')\,\mathrm{d}V(\mathbf{z}), \quad (I = A,B). \tag{29}$$

Eq. (26) gives, for $\mathbf{x} \in V_I$, $(I = A,B)$:

$$\boldsymbol{\sigma}(\mathbf{x})\,|_{V_I} = \mathbf{D}_I : \boldsymbol{\varepsilon}(\mathbf{x}) + \alpha \int_V [\mathbf{D}_A G_A(\mathbf{x},\mathbf{x}') + \mathbf{D}_B G_B(\mathbf{x},\mathbf{x}')] : [\boldsymbol{\varepsilon}(\mathbf{x}') - \boldsymbol{\varepsilon}(\mathbf{x})]\,\mathrm{d}V'. \tag{30}$$

Alternatively, on posing

$$\Gamma_I(\mathbf{x}) := \int_V G_I(\mathbf{x},\mathbf{x}')\,\mathrm{d}V', \quad (I = A,B) \tag{31}$$

and noting that Eq. (24) takes on the form

$$\mathbf{K}(\mathbf{x}) = \mathbf{D}_A \Gamma_A(\mathbf{x}) + \mathbf{D}_B \Gamma_B(\mathbf{x}). \tag{32}$$

Eq. (30) can also be written

$$\boldsymbol{\sigma}(\mathbf{x})\,|_{V_I} = [\mathbf{D}_I - \alpha\mathbf{D}_A\Gamma_A(\mathbf{x}) - \alpha\mathbf{D}_B\Gamma_B(\mathbf{x})] : \boldsymbol{\varepsilon}(\mathbf{x}) + \alpha \int_V [\mathbf{D}_A G_A(\mathbf{x},\mathbf{x}') + \mathbf{D}_B G_B(\mathbf{x},\mathbf{x}')] : \boldsymbol{\varepsilon}(\mathbf{x}')\,\mathrm{d}V', \tag{33}$$

which is equivalent to (25).

From (30), or (33), it is evident that in the case of uniform strain, say $\boldsymbol{\varepsilon}(\mathbf{x}) = \bar{\boldsymbol{\varepsilon}}$ in V, the stress turns out to be piecewise uniform, that is, $\boldsymbol{\sigma}(\mathbf{x}) = \mathbf{D}_I : \bar{\boldsymbol{\varepsilon}}\ \forall \mathbf{x} \in V_I, (I = A, B)$, just like if the material was local in nature.

Eq. (33) can be regarded as derived from an Eringen-type model of the form

$$\boldsymbol{\sigma}(\mathbf{x}) = \alpha \int_V [\mathbf{D}_A G_A(\mathbf{x}, \mathbf{x}') + \mathbf{D}_B G_B(\mathbf{x}, \mathbf{x}')] : \boldsymbol{\varepsilon}(\mathbf{x}')\, \mathrm{d}V' \tag{34}$$

(generalization of (1) for piecewise homogeneous materials) through an enhancement procedure based on a stress redistribution concept analogous to that used in Section 2 for deriving (8) from (1). Indeed, the quantity

$$\mathbf{s}(\mathbf{x}', \mathbf{x}) := \alpha[\mathbf{D}_A G_A(\mathbf{x}', \mathbf{x}) + \mathbf{D}_B G_B(\mathbf{x}', \mathbf{x})] : \boldsymbol{\varepsilon}(\mathbf{x}) \tag{35}$$

represents the specific nonlocal stress contribution at point $\mathbf{x}'$ due to the strain $\boldsymbol{\varepsilon}(\mathbf{x})$. The integral of $\mathbf{s}(\mathbf{x}', \mathbf{x})$ with respect to $\mathbf{x}' \in V$ turns out to be, by (31),

$$\int_V \mathbf{s}(\mathbf{x}', \mathbf{x}) \mathrm{d}V' = \alpha[\mathbf{D}_A \Gamma_A(\mathbf{x}) + \mathbf{D}_B \Gamma_B(\mathbf{x})] : \boldsymbol{\varepsilon}(\mathbf{x}). \tag{36}$$

Phenomenologically, this implies that—through the long distance redistribution processes—the local stress $\mathbf{D}_I : \boldsymbol{\varepsilon}(\mathbf{x})$, $(I = A, B)$, redistributes completely within V at points $\mathbf{x} \in V$ where $\alpha[\mathbf{D}_A \Gamma_A(\mathbf{x}) + \mathbf{D}_B \Gamma_B(\mathbf{x})] = \mathbf{D}_I$, but in general a fraction of it, equal to $[\mathbf{D}_I - \alpha \mathbf{D}_A \Gamma_A(\mathbf{x}) - \mathbf{D}_B \Gamma_B(\mathbf{x})] : \boldsymbol{\varepsilon}(\mathbf{x})$ cannot be redistributed and remains attached as local stress to $\mathbf{x}$. Eq. (34) is therefore to be replaced by the two-component local/nonlocal model (33), which accounts for the macroscopic boundary surface effects of either $S = \partial V$ and S_{AB}.

The results obtained previously in this section can be straightforwardly extended to the case of a domain V with any number of subdomains, say $V = V_A \cup V_B \cup \cdots \cup V_M$, each subdomain being homogeneous with moduli tensor $\mathbf{D}_I$ constant. Repeating the reasoning developed previously, one easily obtains the extended forms of Eqs. (28), (30), (32) and (33) as follows:

$$\mathbf{k}(\mathbf{x}, \mathbf{x}') = \sum_{J=A}^{M} \mathbf{D}_J G_J(\mathbf{x}, \mathbf{x}'), \tag{37}$$

$$\mathbf{K}(\mathbf{x}) = \sum_{J=A}^{M} \mathbf{D}_J \Gamma_J(\mathbf{x}), \tag{38}$$

$$\boldsymbol{\sigma}(\mathbf{x})|_{V_I} = \mathbf{D}_I : \boldsymbol{\varepsilon}(\mathbf{x}) + \alpha \int_V \left[\sum_{J=A}^{M} \mathbf{D}_J G_J(\mathbf{x}, \mathbf{x}') \right] : [\boldsymbol{\varepsilon}(\mathbf{x}') - \boldsymbol{\varepsilon}(\mathbf{x})]\, \mathrm{d}V' \tag{39}$$

or also

$$\boldsymbol{\sigma}(\mathbf{x})|_{V_I} = \left[\mathbf{D}_I - \alpha \sum_{J=A}^{M} \mathbf{D}_J \Gamma_J(\mathbf{x}) \right] : \boldsymbol{\varepsilon}(\mathbf{x}) + \alpha \int_V \left[\sum_{J=A}^{M} \mathbf{D}_J G_J(\mathbf{x}, \mathbf{x}') \right] : \boldsymbol{\varepsilon}(\mathbf{x}')\, \mathrm{d}V', \tag{40}$$

where $G_I(\mathbf{x}, \mathbf{x}')$ and $\Gamma_I(\mathbf{x})$ are defined as in (29) and (31).

Note that in every core subdomain, that is at points $\mathbf{x} \in V_I$ the distance of which is larger than the influence distance of G_I, it is $\Gamma_I(\mathbf{x}) = 1$, whereas $\Gamma_J(\mathbf{x}) = 0\ \forall J \neq I$ and $G_J(\mathbf{x}, \mathbf{x}') = 0\ \forall \mathbf{x}' \in V$ and $\forall J \neq I$. Then, correspondingly, Eq. (40) simplifies as in (12), but with $\mathbf{D}_0$ and G replaced by $\mathbf{D}_I$ and G_I. This means that (40), like (11), can also be considered as an improved form of the Vermeer–Brinkgreve model, extended to piecewise homogeneous materials.

5. Attenuation effects due to inhomogeneity

The inhomogeneity as source of additional attenuation effects is addressed in this section.

5.1. General

In a domain V of nonlocal elastic material, in which the latter is supposed to undergo continuing modulus degradation within a subdomain, say $V_0 \subset V$, the long distance particle interaction forces between points $\mathbf{x} \in V \setminus V_0$ with those in V_0 are nonvanishing in any intermediate degradation state of V_0, but vanishing in the limit failure condition for the material in V_0. This fact, as previously stated, makes it reasonable to conjecture that a nonlocal material (macroscopic) inhomogeneity may be the source of further attenuation of the long distance particle interaction forces, likely the larger, the higher the inhomogeneity.

In the case of homogeneous material, the influence function $g(\mathbf{x}, \mathbf{x}') = \bar{g}(r/\ell)$ accounts for the attenuation effects through the Euclidean (or geodetical) distance $r(\mathbf{x}, \mathbf{x}')$. In the case of inhomogeneous material, in the absence of experimental data, the larger attenuation effects can be macroscopically accounted for making use of some equivalent distance, say $r_{\mathrm{eq}}(\mathbf{x}, \mathbf{x}')$, such that $r_{\mathrm{eq}}(\mathbf{x}, \mathbf{x}') \geqslant r(\mathbf{x}, \mathbf{x}')\ \forall(\mathbf{x}, \mathbf{x}') \in V$, and thus $\bar{g}(r_{\mathrm{eq}}/\ell) \leqslant \bar{g}(r/\ell)$, and in particular $r_{\mathrm{eq}}(\mathbf{x}, \mathbf{x}') \to \infty$, hence $g(r_{\mathrm{eq}}) \to 0$, in the case of complete attenuation. An equivalent distance like that is defined hereafter.

With reference to the piecewise homogeneous material of Section 4, let $\Pi(\mathbf{x}, \mathbf{x}')$ denote an (oriented) path from $\mathbf{x}$ to $\mathbf{x}'$ in V, not intersecting the external boundary surface ∂V, and let $p(\mathbf{x}, \mathbf{x}')$ be its length. If $\Pi(\mathbf{x}, \mathbf{x}')$ intersects the internal boundary surface that separates the subdomains V_I from one another, then $p(\mathbf{x}, \mathbf{x}')$ is to be suitably augmented by a fictitious distance simulating the further attenuation of the long distance nteraction forces (attenuation effects), caused by the inhomogeneities. For this purpose, the inhomogeneities can be thought of as attenuation sources lumped at the intersection points of $\Pi(\mathbf{x}, \mathbf{x}')$ with the internal boundary, say $\mathbf{x}_k$, $(k = 1, 2, \ldots, m)$, having curvilinear abscissas s_k over the oriented path $\Pi(\mathbf{x}, \mathbf{x}')$.

Every attenuation source is to be associated with the modulus jump at the related intersection point. Denoting by η some scalar measure of the moduli tensor $\mathbf{D}$, such that $\eta = 0$ if, and only if, $\mathbf{D} = \mathbf{0}$, (for instance $\eta = (D_{ij}D_{ij})^{1/2}/d$, where d is an adimensionalizing constant), the augmented distance $p_{\mathrm{a}}(\mathbf{x}, \mathbf{x}')$ is expressed as

$$p_{\mathrm{a}}(\mathbf{x}, \mathbf{x}') := \theta p(\mathbf{x}, \mathbf{x}'), \tag{41}$$

where the *attenuation factor* $\theta \geqslant 1$ is a functional of the path $\Pi(\mathbf{x}, \mathbf{x}')$ to which it is associated. Hypothetically, θ is defined as

$$\theta = \theta[\Pi(\mathbf{x}, \mathbf{x}')] := 1 + \sum_{k=1}^{m} C_k \frac{|\ \eta_k^+ - \eta_k^-\ |}{\sqrt{\eta_k^+ \eta_k^-}}, \tag{42}$$

where $\eta_k^+ := \eta(s_k + 0)$ and $\eta_k^- := \eta(s_k - 0)$ are the side limit values of $\eta(s)$ at s_k and $|\ \eta_k^+ - \eta_k^-\ |$ the related jump; also, the (adimensional) coefficients C_k are a set of constants that characterize the internal boundary surface as for its attenuation capabilities at points $\mathbf{x}_k$. These capabilities of the internal boundary surface can be represented by means of a surface coefficient, say C, for which no experimental indications are available, but it can be argued that C is in general inhomogeneous and that—at a fixed point—may depend on the modulus jump at the same point.

The equivalent distance between the fixed points $\mathbf{x}$, $\mathbf{x}'$ in V is assumed equal to the minimum value of $p_{\mathrm{a}}(\mathbf{x}, \mathbf{x}')$ with respect to all paths $\Pi(\mathbf{x}, \mathbf{x}')$; formally

$$r_{\mathrm{eq}}(\mathbf{x}, \mathbf{x}') := \min_{\{\Pi(\mathbf{x}, \mathbf{x}')\}} p_{\mathrm{a}}(\mathbf{x}, \mathbf{x}'), \tag{43}$$

which cannot be smaller than the geodetical distance, $r(\mathbf{x},\mathbf{x}')$, given by

$$r(\mathbf{x},\mathbf{x}') := \min_{\{\Pi(\mathbf{x},\mathbf{x}')\}} p(\mathbf{x},\mathbf{x}'). \tag{44}$$

The minimum operation (43) provides r_{eq} as the length r of an (in general unique) optimal path, amplified by the related attenuation factor, namely $r_{\mathrm{eq}}(\mathbf{x},\mathbf{x}') = \theta r(\mathbf{x},\mathbf{x}')$.

Note that, since $p_{\mathrm{a}}(\mathbf{x},\mathbf{x}')$ and $r_{\mathrm{eq}}(\mathbf{x},\mathbf{x}')$ are independent of the path orientation, then both $p_{\mathrm{a}}(\mathbf{x},\mathbf{x}')$ and $r_{\mathrm{eq}}(\mathbf{x},\mathbf{x}')$ are symmetric, i.e. $r_{\mathrm{eq}}(\mathbf{x},\mathbf{x}') = r_{\mathrm{eq}}(\mathbf{x}',\mathbf{x})$.

The influence function g as a function of r_{eq}, i.e. $g = \bar{g}(r_{\mathrm{eq}}/\ell)$, exhibits the same shape as $g = \bar{g}(r/\ell)$ related to homogeneous material, Fig. 2(a) and (b), whereas as a function of the length r of the optimal path, that is $g = \bar{g}(\theta r/\ell)$, exhibits a shape depending, through θ, on the modulus jumps located along the optimal path. Considering, for instance, a piecewise homogeneous one-dimensional (infinite) domain with a single modulus jump at x_0, and $\eta_A = 1$ for $x < x_0$, $\eta_B \leqslant 1$ for $x > x_0$, the equivalent distance from a point x such that $0 < x_0 - x < R$, (Fig. 3), is $r_{\mathrm{eq}} = \theta \mid x' - x \mid$, where $\theta = 1$ for $x' < x_0$, but $\theta = 1 + C \mid 1 - \eta_B \mid / \sqrt{\eta_B}$ for $x' > x_0$. Thus, the shape of the influence curve centered at x depends on the η_B value as depicted in Fig. 3, where the curve branch $x' > x_0$ is shown to be continuous and to lower with decreasing η_B: at the limit for $\eta_B \to 0$, it tends to the flat pattern $\bar{g} = 0$, what reflects the fact that correspondingly the point x_0 becomes an external boundary point of the region $x < x_0$, hence no interaction is allowed to occur between points $x < x_0$ and points $x > x_0$.

The effects of the inhomogeneities upon the functions $G_I(\mathbf{x},\mathbf{x}')$, $\Gamma_I(\mathbf{x})$ can be discussed in an analogous way as for g. For instance, the plots of $\Gamma_A(\mathbf{x})$, $\Gamma_B(\mathbf{x})$ for a piecewise homogeneous one-dimensional domain $0 \leqslant x \leqslant L$ with $\eta_A = 1$ for $0 \leqslant x < L/2$ and $\eta_B \leqslant 1$ for $L/2 < x \leqslant L$, are depicted in Fig. 4 for a few η_B values and for the bi-exponential function as primitive influence function.

5.2. A few typical cases

Hereafter, a few typical cases are discussed in order to verify whether the equivalent distance as defined previously may constitute an effective analytical tool capable to correctly account—at least qualitatively—for the macroscopic attenuation effects in the material system, even in a limit condition in which one of the subdomains degrades till complete failure.

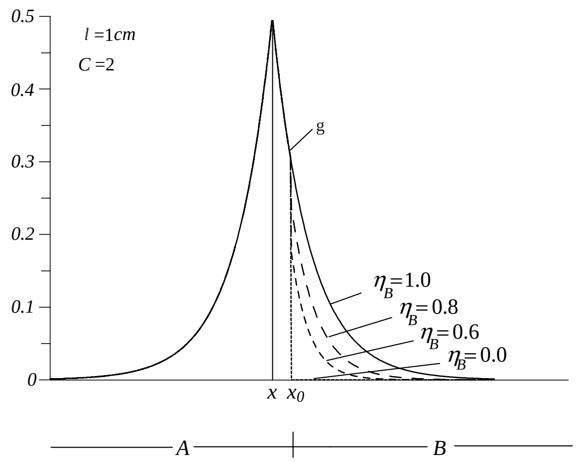

Fig. 3. Bi-exponential influence function $g = (1/2\ell)\exp(-\theta r)/\ell)$ in a piecewise homogeneous one-dimensional (infinite) domain with modulus jump at x_0, for different values of the modulus η_B in the region B: $x > x_0$.

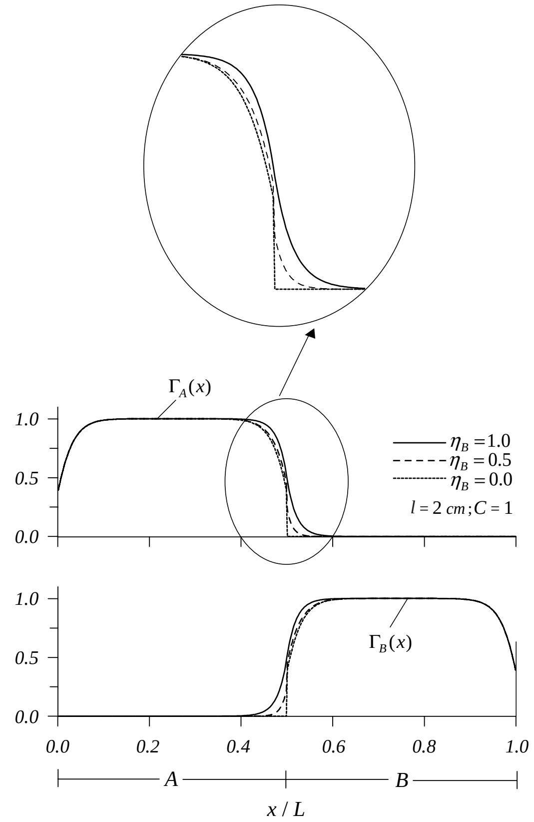

Fig. 4. Plots of the functions $\Gamma_A(x)$ and $\Gamma_B(x)$ for a piecewise homogeneous bar of length L with the modulus jump in the middle section, relative to the primitive and weighed influence functions depicted in Fig. 2.

In Fig. 5(a), the (flat) internal boundary surface S_{AB} devides V into two (simply connected) subdomains, $V = V_A \cup V_B$, each being homogeneous. For more generality, a decohesion crack is supposed to exist in S_{AB}. In order to apply (43) to a pair of fixed points $\mathbf{x} \in V_A$ and $\mathbf{x}' \in V_B$, it is sufficient to consider only paths $\Pi(\mathbf{x}, \mathbf{x}')$ everyone intersecting S_{AB} at a single point. Then, by (42) one can write

$$\theta = 1 + C\frac{|\eta_A - \eta_B|}{\sqrt{\eta_A \eta_B}}, \tag{45}$$

which—for C considered homogeneous—remains constant for all such paths $\Pi(\mathbf{x}, \mathbf{x}')$. Therefore, one has from (43):

$$r_{\mathrm{eq}}(\mathbf{x}, \mathbf{x}') = \left[1 + C\frac{|\eta_A - \eta_B|}{\sqrt{\eta_A \eta_B}}\right] r(\mathbf{x}, \mathbf{x}'), \tag{46}$$

where $r(\mathbf{x}, \mathbf{x}')$ is the geodetical distance (44), (coinciding with the Euclidean distance for the pair $(\mathbf{x}, \mathbf{x}')$ on the left of Fig. 5(a)).

Always with reference to Fig. 5(a), note that the attenuation factor (45) depends on the ratio η_A/η_B. In the case $\eta_A = \eta_B$, it is $\theta = 1$, in which case $r_{\mathrm{eq}}(\mathbf{x}, \mathbf{x}') = r(\mathbf{x}, \mathbf{x}')$ for any pair $(\mathbf{x}, \mathbf{x}')$ in the homogeneous V. Also, on letting $\eta_B \to 0$ while η_A remains fixed, then $\theta \to \infty$, $r_{\mathrm{eq}}(\mathbf{x}, \mathbf{x}') \to \infty$, hence $g(\mathbf{x}, \mathbf{x}') = \bar{g}(r_{\mathrm{eq}}/\ell) \to 0$ $\forall \mathbf{x} \in V_A$ and $\forall \mathbf{x}' \in V_B$; that is—since, at the limit, S_{AB} becomes an external boundary surface—no interac-

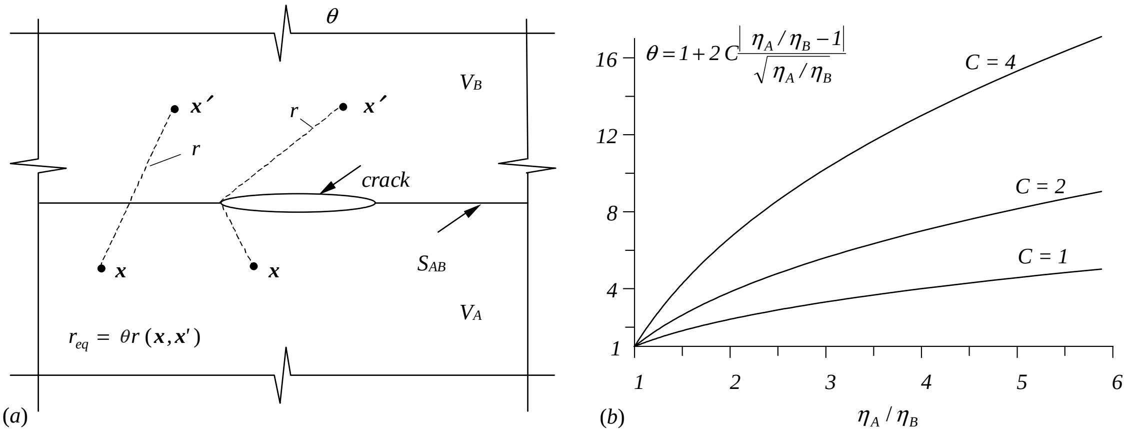

Fig. 5. Piecewise homogeneous domain $V = V_A \cup V_B$ with a flat internal surface S_{AB}: (a) geometrical sketch; (b) plots of the attenuation factor θ as function of the ratio η_A/η_B of the elastic moduli for different values of the surface coefficient C.

tions are allowed to occur between points $\mathbf{x} \in V_A$ and points $\mathbf{x}'$ external to it. The plots of Fig. 5(b) represent each the attenuation factor (45) as a function of the ratio η_A/η_B varying in the interval $1 \leqslant \eta_A/\eta_B < \infty$ and for a particular C value.

In Fig. 6(a), again $V = V_A \cup V_B$, but V_B is thought of as a (convex) inclusion. A pair $(\mathbf{x}, \mathbf{x}') \in V_A$ such that the straight path $\Pi_0(\mathbf{x}, \mathbf{x}')$ (whose length is $r_0(\mathbf{x}, \mathbf{x}') = |\mathbf{x}' - \mathbf{x}|$) intersects S_{AB} is considered. In order to apply (43), it is sufficient considering only two classes of paths $\Pi(\mathbf{x}, \mathbf{x}')$, that is, the paths $\Pi_{\text{in}}(\mathbf{x}, \mathbf{x}')$ intersecting S_{AB} into two points only, and those, $\Pi_{\text{cv}}(\mathbf{x}, \mathbf{x}')$, that circumvent the inclusion. Then, one has from (41) and (42):

$$p_{\text{a}}(\mathbf{x}, \mathbf{x}') = \left[1 + 2C\frac{|\,\eta_A - \eta_B\,|}{\sqrt{\eta_A \eta_B}}\right] p(\mathbf{x}, \mathbf{x}') \tag{47}$$

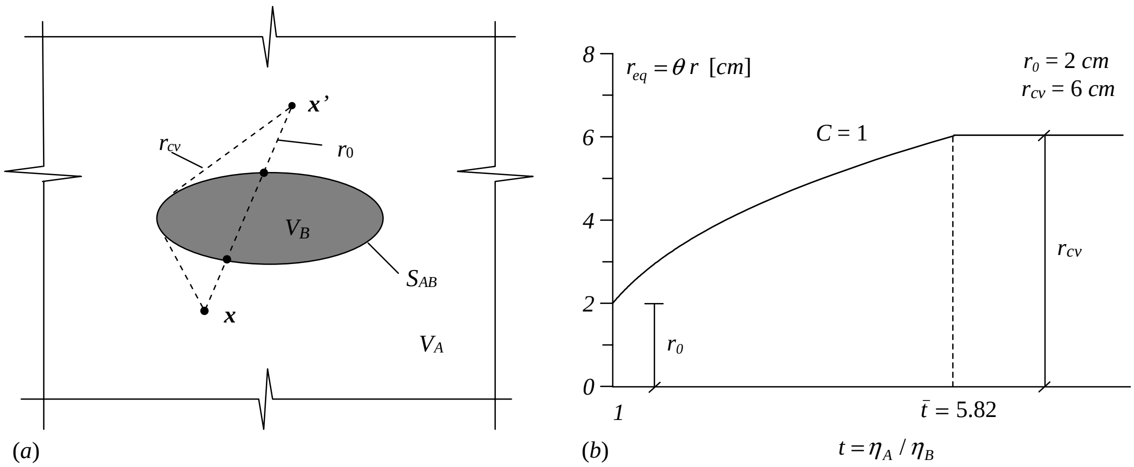

Fig. 6. Homogeneous domain V_A with an (homogeneous) inclusion V_B: (a) geometrical sketch; (b) plot of the equivalent distance r_{eq} as a function of the ratio $t = \eta_A/\eta_B$ of the elastic moduli for a surface coefficient $C = 1$.

for any path, $\Pi_{\rm in}(\mathbf{x},\mathbf{x}')$, intersecting S_{AB} ($m=2$, $C_1=C_2=C$ assumed constant), but

$$p_{\rm a}(\mathbf{x},\mathbf{x}') = p(\mathbf{x},\mathbf{x}') \tag{48}$$

for any path, $\Pi_{\rm cv}(\mathbf{x},\mathbf{x}')$, circumventing the inclusion. Obviously the minimum of $p_{\rm a}(\mathbf{x},\mathbf{x}')$ of (47), or (48), in the set $\{\Pi_{\rm cv}(\mathbf{x},\mathbf{x}')\}$ coincides with the geodetical distance $r_{\rm cv}(\mathbf{x},\mathbf{x}')$ when V_B is considered as a cavity ($\eta_B=0$), Fig. 6(a). It follows that the equivalent distance is

$$r_{\rm eq}(\mathbf{x},\mathbf{x}') = \left[1 + 2C\frac{\mid \eta_A - \eta_B \mid}{\sqrt{\eta_A\eta_B}}\right] r_0(\mathbf{x},\mathbf{x}') \tag{49}$$

whenever $\theta r_0 < r_{\rm cv}$, that is

$$\frac{\mid \eta_A - \eta_B \mid}{\sqrt{\eta_A\eta_B}} < h := \frac{1}{2C}\left(\frac{r_{\rm cv}(\mathbf{x},\mathbf{x}')}{r_0(\mathbf{x},\mathbf{x}')} - 1\right), \tag{50}$$

but

$$r_{\rm eq}(\mathbf{x},\mathbf{x}') = r_{\rm cv}(\mathbf{x},\mathbf{x}') \tag{51}$$

whenever (50) is not satisfied.

The plot of Fig. 6(b) represents $r_{\rm eq}(\mathbf{x},\mathbf{x}')$ of (49) and (51) as a function of the ratio $t=\eta_A/\eta_B$ varying in the interval $1 \leqslant t < \infty$, which is obtained by letting $\eta_B \to 0$ while η_A is taken fixed. The equivalent distance increases monotonically from r_0 to $r_{\rm cv}$ for η_B decreasing from the η_A value ($t=1$) to some $\bar{\eta}_B$, ($t=\bar{t}=\eta_A/\bar{\eta}_B$), but remains constant and equal to $r_{\rm cv}(\mathbf{x},\mathbf{x}')$ for η_B smaller than $\bar{\eta}_B$ ($t>\bar{t}$). The $\bar{\eta}_B$ value can be easily found by writing (50) as an equality and then solving the quadratic equation

$$(t-1)^2 - h^2 t = 0. \tag{52}$$

This admits two positive roots, one ($\bar{t}$) greater, the other ($\bar{\bar{t}}$) smaller, than unity, that is:

$$\bar{t} := \frac{\eta_A}{\bar{\eta}_B} = 1 + \frac{1}{2}h^2 + h\sqrt{1+\frac{1}{4}h^2} > 1, \quad (\bar{\eta}_B < \eta_A), \tag{53}$$

$$\bar{\bar{t}} := \frac{\bar{\bar{\eta}}_B}{\eta_A} = 1 + \frac{1}{2}h^2 - h\sqrt{1+\frac{1}{4}h^2} < 1, \quad (\bar{\bar{\eta}}_B > \eta_A). \tag{54}$$

It is worth noting that—always in the hypothesis that C is homogeneous—the equivalent distance $r_{\rm eq}(\mathbf{x},\mathbf{x}')$ under discussion, Fig. 6(b), is independent of the size and shape of the inclusion as far as the inclusion is sufficiently stiff ($\eta_B > \bar{\eta}_B$, $t<\bar{t}$), whereas for smaller values of η_B ($t>\bar{t}$), $r_{\rm eq}=r_{\rm cv}$ depends on the size and shape of the inclusion. The case envisioned here is similar to the one encountered in practice when a narrow region of a body degrades with increasing the loading till the formation of a crack with surface separation. Also, note that were the inclusion V_B of Fig. 6(a) shaped as a layer traversing the whole body V, then $r_{\rm cv}=\infty$, $h=\infty$, hence Eq. (49) would be always valid.

The attenuation effects due to the inhomogeneity imply that, in a domain, $V=V_A\cup V_B$, Fig. 7(a), at points $\mathbf{x}\in V_A$ sufficiently close to the internal boundary S_{AB}, the influence sphere $\Sigma(\mathbf{x})$ is modified in the part exceeding S_{AB}. In fact, the latter part of $\Sigma(\mathbf{x})$ is substituted by a cup, locus of points $\mathbf{x}$ (N in Fig. 7(a)) such that $\theta \mid \mathbf{x}'-\mathbf{x} \mid = R$, where $\theta = 1 + C \mid \eta_A - \eta_B \mid / \sqrt{\eta_A\eta_B}$. Since θ increases with decreasing η_B while η_A is fixed, and $\theta \to \infty$ for $\eta_B \to 0$, also considered that $\mathbf{x}'$ has to lie in V_B, follows that at the limit the influence region coincides with $\Sigma(\mathbf{x})$ deprived by its part external to S_{AB}, Fig. 7(b).

Note that, in the limit condition depicted in Fig. 7(b), where $\eta_B=0$, the equivalent distance between two points of V_B, say $\mathbf{y}$ and $\mathbf{y}'$, equals the Euclidean distance $\mid \mathbf{y}'-\mathbf{y} \mid$, hence $g(\mathbf{y},\mathbf{y}')>0$; however the stress $\boldsymbol{\sigma}(\mathbf{y})$

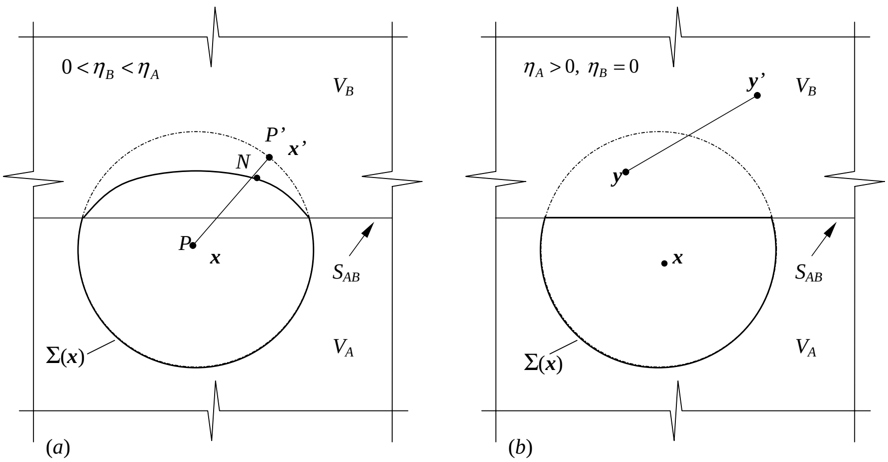

Fig. 7. Modification of the influence sphere $\Sigma(\mathbf{x})$ in the vicinity of the internal boundary surface S_{AB} in a piecewise homogeneous domain $V = V_A \cup V_B$: (a) intermediate condition in which the elastic modulus η_B satisfies $0 < \eta_B < \eta_A$; (b) limit condition in which $\eta_A > 0$, $\eta_B = 0$.

predicted by the material model is identically vanishing, as it has to be. In fact, by Eq. (33), one can write with reference to Fig. 7(b):

$$\boldsymbol{\sigma}(\mathbf{y})\,|_{V_B} = [\mathbf{D}_B - \alpha\mathbf{D}_A\Gamma_A(\mathbf{y}) - \alpha\mathbf{D}_B\Gamma_B(\mathbf{y})] : \boldsymbol{\varepsilon}(\mathbf{y}) + \alpha\int_V [\mathbf{D}_A G_A(\mathbf{y},\mathbf{x}') + \mathbf{D}_B G_B(\mathbf{y},\mathbf{x}')] : \boldsymbol{\varepsilon}(\mathbf{x}')\,\mathrm{d}V'. \tag{55}$$

Since, in the limit condition under consideration, it is $\mathbf{D}_B = \mathbf{0}$, $g(\mathbf{y},\mathbf{z}) = 0\ \forall\mathbf{z} \in V_A$, hence by (29) and (31), $\Gamma_A(\mathbf{y}) = 0$, $G_A(\mathbf{y},\mathbf{x}') = 0\ \forall\mathbf{x}' \in V$, it follows that $\boldsymbol{\sigma}(\mathbf{y}) = 0\ \forall\mathbf{y} \in V_B$.

Other special cases may still be considered, but those previously discussed seem to be sufficient to assess the effectiveness of the equivalent distance as a means to account—through the attenuation function $g(\mathbf{x},\mathbf{x}') = \bar{g}(r_{\mathrm{eq}}/\ell)$—for the macroscopic long distance interactions in a nonlocal inhomogeneous material, at least as for their qualitative features.

This section is concluded observing that several concepts of the present theory and shown by the sketches of Figs. 4 and 5 are surprisingly similar to concepts of common use within a computational mechanics context as the moving least-square reproducing kernel methods (Liu et al., 1997) and the meshless methods (Belytschko et al., 1996). Since these methods make use of an integral formula with a kernel having a finite support like in the nonlocal constitutive model, similar (macroscopic) boundary surface effects arise. As a matter of facts, the concepts of "visibility" and "diffraction" used in the computational mechanics context (Belytschko et al., 1996) are equivalent to the concept of "geodetical distance" employed in the context of nonlocal elasticity (Polizzotto, 2001), as well as to the more general concept of "equivalent distance" herein proposed.

6. Numerical application

An elastic bar of length L and unit cross-section, clamped at the end section $x = 0$ and subjected to a given displacement $\bar{u}$ at the end section $x = L$, is considered. The bar is piecewise homogeneous, with Young modulus $E(x) = \eta E_0$ for $0 \leqslant x < L/2$ and $E(x) = E_0$ for $L/2 < x \leqslant L$, Fig. 8(a). The one-dimensional version of Eq. (33) has been employed with a bi-exponential influence function, i.e. $\bar{g}(r_{\mathrm{eq}}/\ell) = (1/2\ell)\exp(-r_{\mathrm{eq}}/\ell)$,

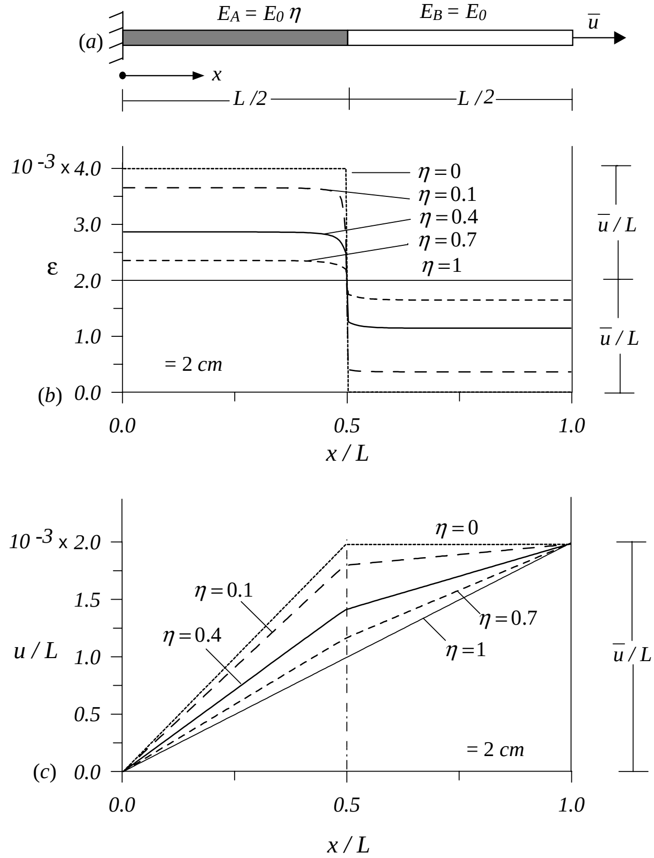

Fig. 8. Piecewise homogeneous bar with the left end fixed, subjected to a displacement $\bar{u} = 0.2$ cm at the right end, and having different Young moduli in the two half portions: (a) geometrical sketch; (b) strain profiles for different values of $\eta \leqslant 1$; (c) corresponding displacement responses. (Corresponding stress values in Table 1.)

$\ell = 2$ cm, $R = 12$ cm, used as a primitive influence function. The data are: $E_0 = 21 \times 10^4$ MPa, $L = 100$ cm, $\bar{u} = 0.2$ cm.

Since the stress σ is uniform by equilibrium, the integral equation to solve can be written as

$$[1 - \alpha\Phi(x)]\omega(x) - \alpha \int_0^l F(x,x')\omega(x')\,\mathrm{d}x' = 1, \quad (0 \leqslant x \leqslant L), \tag{56}$$

where

$$F(x,x') := \eta G_A(x,x') + G_B(x,x'), \tag{57}$$

$$\Phi(x) := \eta\Gamma_A(x) + \Gamma_B(x), \tag{58}$$

$$\omega(x) := \frac{E_0}{\sigma}\varepsilon(x). \tag{59}$$

Eq. (56) is a Fredholm integral equation of second kind, which is solved numerically to obtain $\omega(x)$. Then, by the boundary condition $u(L) = \bar{u}$, one has from (59):

Table 1
Stress responses of the bar in Fig. 8

η	Stress σ [MPa]
1.0	420.00
0.7	346.06
0.4	240.52
0.1	77.14
0.0	0.0

$$\sigma = \frac{E_0 \bar{u}}{\int_0^L \omega(x)\,\mathrm{d}x}. \tag{60}$$

Eq. (56), solved with $\alpha = -1$, $C = 2$ and with different values of η, provides the strain profiles $\varepsilon(x) = \sigma\omega(x)/E_0$ plotted in Fig. 8(b). The corresponding displacement responses are also reported in Fig. 8(c), whereas the related stress values are shown in Table 1. The following is noted:

(a) For $\eta = 1$, as expected, the solution coincides with the local type solution with $\varepsilon = \bar{u}/L = 2 \times 10^{-3}$ cm and $\sigma = E_0\bar{u}/L = 420$ MPa.
(b) For $\eta = 0$, the solution obtained interprets the limit idealized condition in which the left half bar has vanishing stiffness and undergoes a uniform strain $2\bar{u}/L = 4 \times 10^{-3}$ cm at zero stress, whereas the right half bar displaces rigidly by the imposed amount $\bar{u} = 0.2$ cm. The strain profile is piecewise constant with zero strain in the right half bar.
(c) For intermediate values of η, $0 < \eta < 1$, two extreme bar portions of equal length stretch uniformly at different strain values, whereas in the remaining central portion (the length of which is the shorter, the smaller η) the strain ε varies continuously between the extreme values.

Other negative values of α have been used, with no remarkable differences in the numerical results. With a positive α value, numerical instability problems arise from the middle section discontinuity; such pathological behaviour of the solution procedure emerges also near the end sections making use of the Eringen-type model (34), ($\alpha = 1$). These numerical instability features tend to become modest with ℓ sufficiently small.

7. Conclusions

A two-component local/nonlocal constitutive model for (macroscopically) inhomogeneous elastic materials (with constant internal length) has been proposed, in which the stress is the superposition of the local stress and of a nonlocal contribution functional of the strain difference instead of the strain. Such a model naturally satisfies the condition that the predicted stress equals the local stress whenever the strain field is uniform, as usually required for a phenomenological constitutive elastic model. This nonlocal stress–strain law has been derived as the state equation associated with a suitable Helmholtz free energy for inhomogeneous elastic material in isothermal conditions. The case of major interest addressed here is that of a piecewise homogeneous material with two or more homogeneous subregions. A more general thermodynamic treatment of nonlocal inhomogeneous materials is the subject of an ongoing research.

The concept of equivalent distance, $r_{\mathrm{eq}}(\mathbf{x}, \mathbf{x}')$, has been introduced, which exceeds the Euclidean, or geodetical, distance $r(\mathbf{x}, \mathbf{x}')$, and the excess distance aims at taking into account the attenuation effects due to the inhomogeneities. These inhomogenities are related to the modulus jumps at the points where the path joining $\mathbf{x}$ and $\mathbf{x}'$ intersects the internal boundary and to this purpose a suitable scalar measure of the material moduli tensor has been introduced. In the absence of available experimental data, a definition of $r_{\mathrm{eq}}(\mathbf{x}, \mathbf{x}')$ has been heuristically suggested, which seems suitable for a satisfactory phenomenological description of the material constitutive behaviour, at least from the qualitative point of view. This definition of r_{eq} has been discussed in relation to various typical cases that can occur in practice, but obviously further investigations are necessary.

It has been shown that the proposed nonlocal constitutive model can also be considered derivable from the Eringen nonlocal elasticity model through a symmetry-saving enhancement procedure. Indeed, a redistribution process of the local stress has been envisioned as a macroscopic long distance phenomenon promoted by nonlocality; at the points, likely close to the boundary surface, where this stress redistribution cannot be complete, the nonredistributed local stress has to be considered as a *correction local-type constitutive component* to be added to the nonlocal one at the respective points.

The proposed constitutive model possesses, besides the elastic moduli and the internal length ℓ, an additional material constant (α), which controls the volume fractions of the local/nonlocal constitutive components. It has been suggested to choose this α in such a way that the solution procedure be numerically stable. More general considerations based on the convexity of the Helmholtz free energy potential in a suitable functional space should be possible in order to assess an interval of admissible values for α, but this point has been left open to future research work.

The model has a strong formal similarity with the Vermeer and Brinkgreve (1994) model, of which it can be considered an improved version extended to inhomogeneous materials. In the case of homogeneous material, and at points located at a distance from the boundary surface larger than the influence distance, the proposed model coincides with the Vermeer–Brinkgreve's with α as the related characteristic material constant.

A piecewise homogeneous bar structure, fixed at one end and subjected to a given displacement at the other end, has been addressed, for which the numerical solution procedure exhibits no pathological behaviour, provided the α coefficient is taken with a negative value. Different decreasing values of the Young modulus have been considered in the half bar adjacent to the fixed end: from the value corresponding to the fully homogeneous bar condition, to the zero value pertaining to the limit condition in which the less stiff half bar is completely degraded and thus the bar reduces to only the other half portion displacing as a rigid body at zero stress. A meaningful solution has been obtained in every degraded bar condition, including the idealized failure condition. This result can be interpreted as an assessment of the consistency of the proposed equivalent distance concept.

A limitation of the present nonlocal constitutive model is the assumption of homogeneous internal length. Extensions to materials with inhomogeneous internal length are of practical interest. Another limitation of the proposed model is that it requires, besides the *primitive* influence function, also a *weighed* influence function. Furthermore, the concept of equivalent distance, having a basic role in the model, turns out to be more elaborated than that of Euclidean, or even geodetical, distance. Hence, the proposed model leads to an increase of the computational burden; however, a judgement over these computational aspects require further numerical investigations with multi-dimensional examples.

All this will be the subject of subsequent research work. Extensions to other constitutive models (plasticity, damage) are under study.

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